

International Managerial Economics: Skills

- In seeking to explain economic phenomena, we typically employ models – simplified descriptions of reality that enable us to make predictions.
- In analysing these problems, we frequently wish to quantify them, that is, provide mathematical (and/or graphical) representations of objectives, constraints, and outcomes. This includes describing economic quantities as a mathematical function of one or more variables, e.g.:
 - A firm's costs as a function of output
 - Market demand for a good in terms of price
 - A consumer's utility (satisfaction) in terms of quantities of different goods purchased.
- So, if we are modelling a firm's output decision, we would need to include:
 - A function expressing the firm's costs as a function of output
 - A function expressing demand for the firm's good in terms of price.
- We could then use these equations, along with some basic mathematical techniques, to calculate the firm's optimal level of output.

Functions

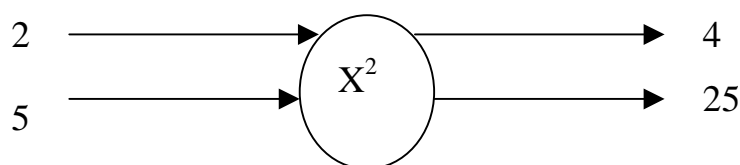
A function is a rule that takes one or more numbers as inputs (arguments) and gives another number as output.

e.g. $f(x) = x^2$

Here, “f” is the name given to the function, x represents the input, x^2 tells us how to calculate the answer in terms of the input.

e.g. $f(2) = 2^2 = 2 \times 2 = 4$, $f(5) = 5^2 = 5 \times 5 = 25$ etc.

A function can be seen as a ‘black box’ for converting numbers into other numbers:



If we want to write another variable as a function of x, we may write $y = f(x)$, or just (in this case) $y = x^2$.

E.g. we could express a demand function by

$$Q = 20 - 5P$$

Where Q is quantity demanded and P is price. Thus quantity is expressed as a function of price.

Functions of more than one variable

We may have functions of two or more variables. For example

$$F(x,y) = xy + 2x$$

This function requires two numbers input to produce an answer. For example,

$$F(4,5) = 4*5 + 2*4 = 28$$

Here x is 4 and y is 5.

For example in economics, a firm's output may depend on their input of Labour (L) and Capital (K), for example:

$$Q = 100L^{0.5}K^{0.5}$$

Where Q is units of output, L is number of workers and K is, perhaps, number of machines.

For example, if L=9 and K=16, then

$$Q = 100*(9^{0.5})*(16^{0.5}) = 100*3*4 = 1,200 \text{ units.}$$

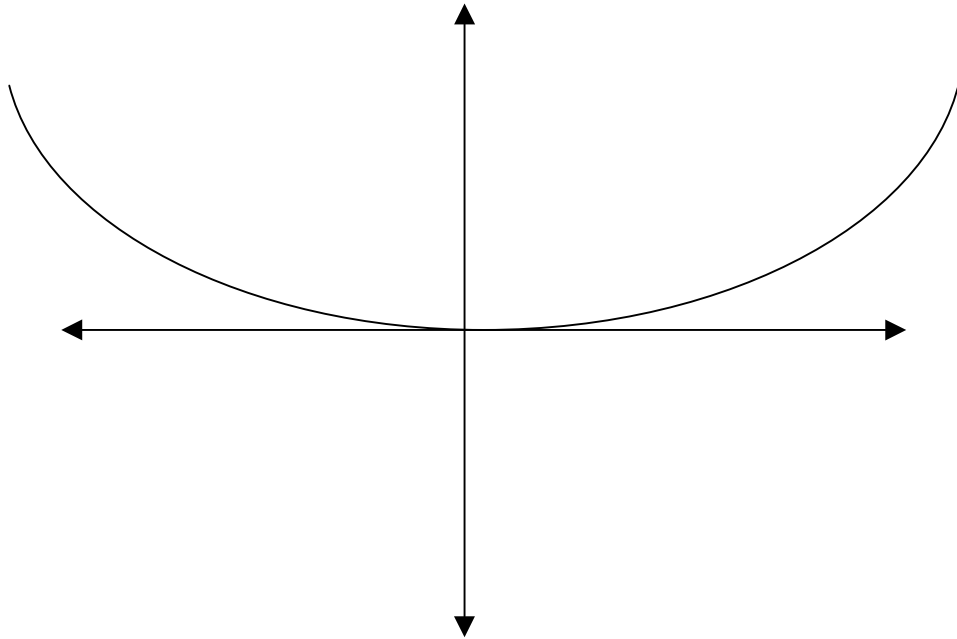
(Since raising to the power 0.5 is the same as taking the square root).

Graphs of functions

Functions of 1 variable can easily be represented on a graph. E.g. if we have

$$F(x) = x^2$$

This has a graph something like:



The simplest sorts of functions are linear. These are of the form

$$Y = a + bX$$

Where **a** and **b** are constants, e.g. $Y = 5 - 2X$. The graphs of these functions are straight lines. For example if we have the demand function

$$Q = 20 - 5P$$

First, because we always put price on the vertical axis, we need to get P on to the left hand side of the equation. Thus

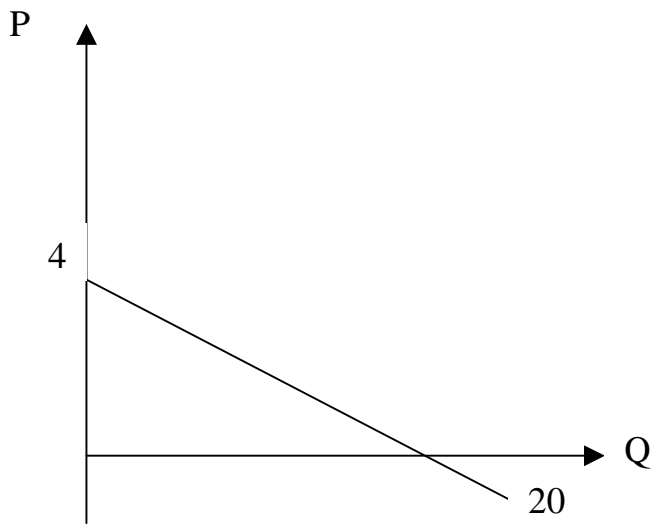
$$Q + 5P = 20$$

$$5P = 20 - Q$$

$$P = (20/5) - (Q/5)$$

$$P = 4 - Q/5$$

This has a graph:



Differentials, slopes, and rates of change

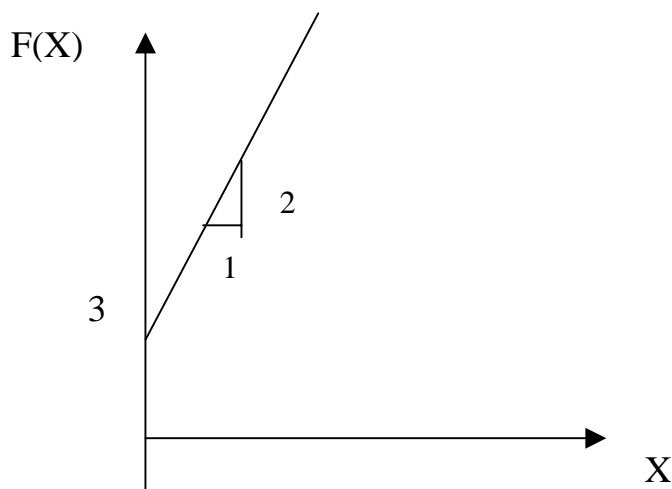
In economics, we are frequently interested in *marginal effects* – Marginal cost, marginal utility, marginal revenue, etc.

This means the additional effect of *one more* unit of something (one more unit of output, one more unit of a good purchased, etc.) – that is, it is the *rate of change* of one variable as another changes.

This is the same as the slope of a function on a graph.

E.g. If we have a function

$F(x) = 3 + 2X$, with the graph



Each increase of 1 in X leads to an increase in 2 of $F(X)$. The slope of the line is 2. We could say that the marginal increase in F for a change in X is 2.

Straight lines are easy, as the slope is always the same – the slope of the above graph is 2 for all values of X .

When we have a non-linear function – for example $F(X) = X^2$ - the slope of the graph, and therefore the rate of change of $F(X)$, varies depending on the value of X .

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We can measure the rate of change, or the marginal effect, in two different ways: first, most easily, by looking at the change in X and the change in $F(X)$ between two points:

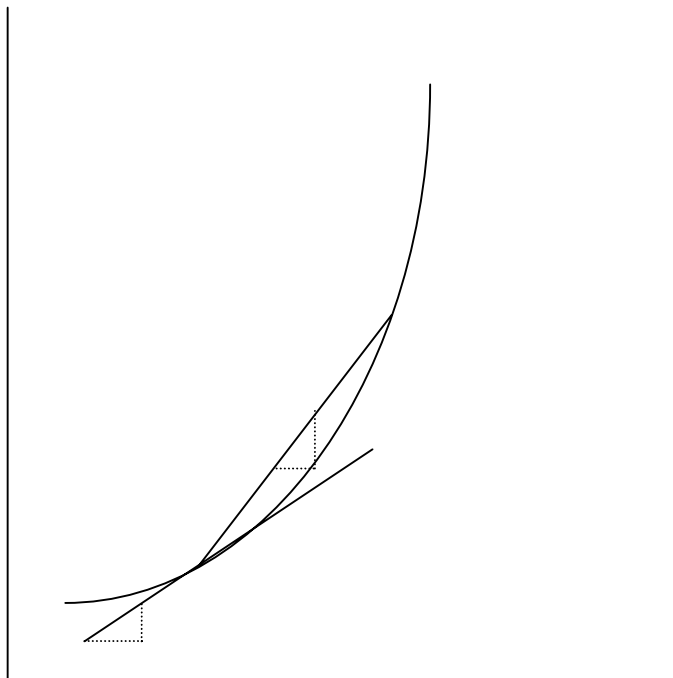
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For example, between $X=2$ and $X=3$, $F(X)$ goes from 4 to 9, so the rate of change is $(9-4)/(3-2) = 5$.

But it is more precise to measure the rate of change *at a particular point*. We do this by looking at the slope of the *tangent* line to the curve at the point we're interested in.

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We can also see that, if we are taking the slope between two points, the nearer these points are together, the closer the slope is to the slope of the tangent line. .



The rate of change at a particular point X – that is, the slope of the tangent line – is also known as the differential of the function $F(x)$ at X .

We write the differential at a point X as $F'(X)$. (For example, at $X=1$, we write the slope $F'(1)$.)

In general, the differential changes as x changes, and so is itself a function of x , written $F'(x)$.

If we have $Y=F(X)$, then we write the differential as $\frac{dY}{dX}$.

When we find the differential of a function, this is called differentiation.

Rules for differentiation

There are some fairly simple rules for differentiating all the basic functions you are likely to meet in this module.

- 1) Constant functions: $F(X) = a$, where a is a constant, e.g. $F(X) = 3$. These are flat, they have slope 0, so $F'(X) = 0$.
- 2) Linear functions: If $F(X) = a + bX$, then $F'(X) = b$. (Linear functions have a constant slope).
- 3) If $F(X) = X^n$, where n is any number (positive or negative, not necessarily an integer (whole number)), then $F'(X) = nX^{n-1}$.
- 4) If $F(X) = \ln(X)$ where \ln is the *natural logarithm*, then $F'(X) = 1/X$.
- 5) If $F(X) = e^X$ (the exponential function), then $F'(X) = e^X$.

For example (case 3), if $Y = X^2$, then $dY/dX = 2X$ – in other words, the slope *increases* as X increases, as we can see from the graph. .

Rules for combining functions

- 1) Addition of functions: If $F(X) = G(X) + H(X)$, then $F'(X) = G'(X) + H'(X)$
- 2) Multiplication by a constant: If a is a constant, then the differential of $aF(X)$ is $aF'(X)$. (E.g. the differential of $2X^2$ is $2*2X = 4X$).
- 3) Multiplication of functions: If $F(X) = G(X)H(X)$ then $F'(X) = G(X)H'(X) + G'(X)H(X)$.
- 4) Division of functions: If $F(X) = G(X)/H(X)$, then $F'(X) =$

$$\frac{H(X)G'(X) - G(X)H'(X)}{(H(X))^2}$$

For example, if $Y = (X+3)(3-2X)$, we let $G(X) = X+3$, and $H(X) = 3-2X$. Then, $G'(X) = 1$, and $H'(X) = -2$.

Thus:

$$dY/dX = (X+3)*(-2) + 1*(3-2X) = -2X - 6 + 3 - 2X = -4X-3.$$

5) Function of a function: If $F(X) = G(H(X))$, then $F'(X) = G'(H(X))H'(X)$

For example, if $F(X) = e^{(x^2)}$, we let $G(.) = e^{(.)}$, and $H(X)=X^2$.

Now $G'(X) = e^X$, so $G'(.) = e^{(.)}$, so $G'(H(X)) = e^{H(X)} = e^{(x^2)}$.

Also $H'(X) = 2X$, so

$$F'(X)=G'(H(X))H'(X) = e^{(x^2)} * 2X.$$

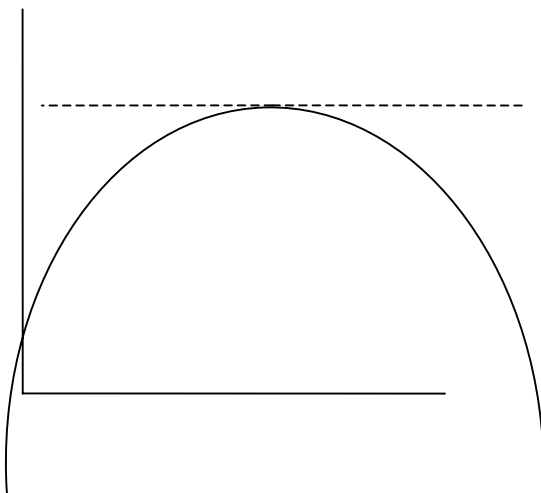
Optimisation in one variable

We are frequently interested in maximising or minimising a quantity, e.g. maximising profits or minimising costs. This can be done using differentiation.

A function is at its maximum or minimum value when it stops rising and starts falling, or vice versa. .

When a function moves from rising to falling (or v.v.), there will be a momentary **stationary point** where it is not changing.

That is, at a **local maximum** or **local minimum** of a function, the differential, $F'(X)$, will be equal to 0 (i.e. the tangent line is flat):



We say a *local* maximum or minimum, because it may not be the *global* highest or lowest point.

Stationary points can also be *points of inflexion*, where the function flattens out then continues in the same direction. .

Example

Suppose a firm faces a demand curve given by:

$$Q = 20 - 3P$$

Where Q is quantity and P is price. How can the firm maximise revenue?

Well, revenue is price*quantity, PQ, which is equal to

$$P(20-3P) = 20P - 3P^2. .$$

$$\text{So, let } F(P) = 20P - 3P^2$$

$$\text{Then } F'(P) = 20 - 6P.$$

A stationary point will come when $F'(P) = 0$, i.e. when

$$20 - 6P = 0$$

Therefore, $20 = 6P$, so

$$P = 20/6 = 3.333$$

At this value, $Q = 20 - 3P = 10$, so revenue = $10 * 3.333 = 33$ and a third.

Classifying stationary points

How can we be sure (apart from the graph) that this is a maximum and not a minimum or a point of inflexion? We do this by looking at the *second differential* – that is, the differential of the differential – which we write $F''(X)$ (or $\frac{d^2Y}{dX^2}$.)

E.g. if $F(X) = X^3$, then $F'(X) = 3X^2$, so $F''(X) = 3 * 2X = 6X$.

This is the rate of change of the rate of change.

Now at a maximum, the rate of change starts positive, goes to zero, then goes negative – so the rate of change is going down, so the rate of change of the rate of change is negative. In other words

If $F''(X) < 0$ at a stationary point, then the point is a local maximum.

The opposite holds at a minimum, so

If $F''(X) > 0$ at a stationary point, the point is a local minimum.

Now in the case of our company, where the revenue function was $F(P) = 20P - 3P^2$, and $F'(P) = 20 - 6P$, with a stationary point at $P=3.333$.

Now $F''(P) = -6$. This is negative at the stationary point (indeed at all values of P), and so the point is a local maximum.

If $F''(X) = 0$ at a stationary point, the point could be a maximum, minimum or point of inflexion. (Look at third, fourth etc. differentials.)

Distinguishing a global maximum or minimum

In general, the global maximum or minimum can occur at any of the local maxima and minima, or at a corner solution – the lowest or highest possible value. (For example, a company's profits may be highest when output is zero.) . It may be necessary to look at all maxima/minima and all possible corner solutions to find the best.

However, there are certain cases where we can be sure a local maximum/minimum is the global maximum/minimum:

If $F''(X) < 0$ for the full range of values a function can take, then any local maximum is the global maximum. (We say such a function is concave).

If $F''(X) > 0$ for the full range of values a function can take, then any local minimum is the global minimum. (We say such a function is convex).

In the case we considered, we found $F''(P) = -6$, which is <0 for all possible values (0 to infinity), so the local maximum we found must be a global maximum.

Example: Marginal costs and marginal revenue

We know that a company maximises profits when marginal costs = marginal revenue. ($MC=MR$). This can be analysed in terms of calculus.

Suppose a company has a Revenue function $R(Q)$, where Q is the output, and a cost function $C(Q)$. Then the profit function, $\Pi(Q)$, can be written $\Pi(Q) = R(Q) - C(Q)$.

Differentiating, $\Pi'(Q) = R'(Q) - C'(Q)$. This will have a stationary point where $\Pi'(Q) = 0$, so $R'(Q) - C'(Q) = 0$, and hence:

$$R'(Q) = C'(Q).$$

But $R'(Q)$ is the rate of change of revenue as output increases, in other words, the marginal revenue. $C'(Q)$ is the rate of change of costs, in other words, the marginal cost. Hence, the equation we have tells us that $MC=MR$.

Optimising functions of more than one variable: partial differentiation and the marginal principal

When we have a function of more than one variable, we can use partial differentiation to find the rate of change of the function with respect to any of the variables.

For example, suppose we have a demand function in terms of price and consumer income given by

$$Q = (20-3P)Y$$

Where Q is quantity demanded, P is price, and Y is average consumer income (in some suitable unit). (In other words, demand is proportional to income.)

We may differentiate this partially, with respect to either P or Y, by treating the other as if it were a constant, and differentiating normally.

This gives $\frac{\partial Q}{\partial P} = -3Y$, and

$$\frac{\partial Q}{\partial Y} = (20-3P).$$

(Note the use of curly d's when we do partial differentiation.)

Simple optimisation in 2 variables is quite similar to one variable:

A stationary point occurs when all partial differentials are equal to zero. This can be a local maximum, a local minimum, or a saddle point.

The rules for deciding which are quite complicated and don't really concern us.

Constrained optimisation

Typically in economics, we face a problem where a consumer or firm must choose how to devote resources between two or more goods or inputs, facing a given price for each, and seeking to maximise some quantity.

A simple example of how this can be dealt with is:

Consider a firm with total cost

$$TC = 4Q_1^2 + 5Q_2^2 - Q_1Q_2$$

And want to find out how to minimise cost with total output equal to 30

That is $Q_1 + Q_2 = 30$

Then to find the best output of Q_2 set $Q_1 = 30 - Q_2$

And substitute

$$\begin{aligned}TC &= 4(30 - Q_2)^2 + 5Q_2^2 - (30 - Q_2)Q_1 \\&= 4(900 - 60Q_2 + Q_2^2) + 5Q_2^2 - 30Q_2 + Q_2^2 \\&= 3600 - 270Q_2 + 10Q_2^2\end{aligned}$$

Then differentiate and set equal to zero in the usual way

$$dTC/dQ_2 = \dots = 0$$

Alternatively can use a technique of Lagrangian multipliers -which is simpler and necessary when there are more constraints, or more complex ones

1. Take the constraint and rewrite as $30 - Q_1 - Q_2$

2. Write a Lagrangian function:

$$L = 4(30 - Q_2)^2 + 5Q_2^2 - (30 - Q_2)Q_1 + \lambda(30 - Q_1 - Q_2)$$

3. Find the unconstrained max or min for this function and you will have the solution to the constrained maximisation problem -the same as above.

$$\delta L / \delta Q_1 = \dots = 0$$

$$\delta L / \delta Q_2 = \dots = 0$$

$$\delta L / \delta \lambda = \dots = 0$$

which we can solve in the usual way.